Probability generating function of a group-arrival queue with set-up time

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Pooja
(MS23MTH009)

under the supervision of

Dr. Sourav Pradhan

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440010 (India)
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Declaration

I, Pooja, hereby declare that this dissertation titled "Probability generating function of a group-arrival queue with set-up time" is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Pooja

Date:

Certificate

This is to certify that the dissertation entitled "**Probability generating function of a group-arrival queue with set-up time**" is a bonafide work done by **Miss Pooja** (Enrollment No. MS23MTH009). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

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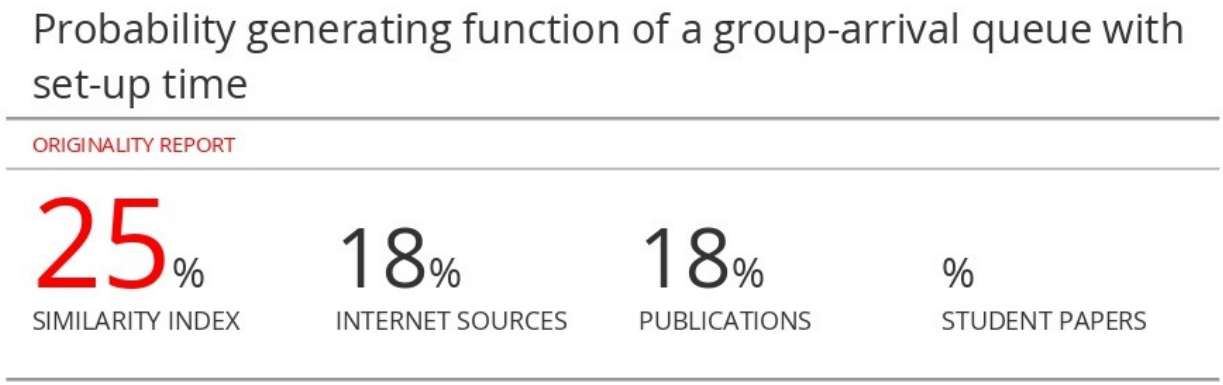

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Pooja

Abstract

Minimization of processing time and associated cost can be accomplished through the employment of batch-service queues. Due to several noteworthy applications in manufacturing system, testing of blood samples, etc., we analyze a single server infinite-buffer size-dependent service time queue. It includes compound poisson process, random serving capacity rule and set-up time. After considering the transition probability matrix, we derive probability generating function of the queue length at departure epoch. We also display probabilities in terms of roots of the denominator.

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Notations

Symbols/Abbreviations	Name
pgf	probability generating function
$\Upsilon^{+}(z)$	pgf of number of customers in the queue at departure epoch
γ_n^{+}	probability that n customers in the queue at departure epoch
$m_i^{(r)}$	probability of i arrivals during the service time of batch-size r

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Chapter 1

Introduction

1.1 Overview of the queueing systems

We all experience waiting in lines in our daily lives at the bank to deposit or withdraw money or in a super market counter or at temple etc. Several questions immediately follow in our mind such as (i) how long will a customer have to wait in a line?, (ii) how many customers will be waiting on an average in the line?, (iii) how many servers should be kept for smooth functioning and less queue?, (iv) in what pattern should the service be provided so that it is most beneficial and cost effective? In order to answer these questions, probabilistic approach for analyzing a queueing system has become an inescapable component of our daily lives. The mathematical study of queueing system, in which customers arrive, wait for service, take service, and then leave the system are very natural and common phenomena. Such queueing phenomena are quite often at the counters of a movie theater, airplanes waiting to take off, traffic jams on the road, even on a telephone call waiting to be put through a receiver etc. Way back to early twentieth century with the works of Erlang on stochastic systems which provided the first apparent need of discussion for queue. The uncertainty regarding customer's arrival time and their service requests can only be captured using the stochastic processes.

In traditional queueing systems, customer's arrival and service occurs individually. However, there exists several situations, where it is noticed that the customers are furnished their requests in batches rather than singly. The main framework of modeling of such scenarios is the batch-service queues. Now a days, such queues are very much essential to improve the utilization of resources in a quite instrumental way. Due to extensive utilization, batch-service queues had been discussed,

where the mean service time of each batch was assumed to be the same without taking care of the effectiveness of batch-size. But such assumption may not hold in numerous real-life applications. Therefore it will be more convincing if it is assumed that the service time depends on the size of the batches. Such queues are generally known as size-dependent service queues and are inevitable due to their widespread applications in testing of blood samples at medical facility center or in the manufacturing system to pick out the defective items. The application of such queue even further stretches to recreational devices at amusement park, wireless telecommunication system for transmitting data, messages, videos, files etc.

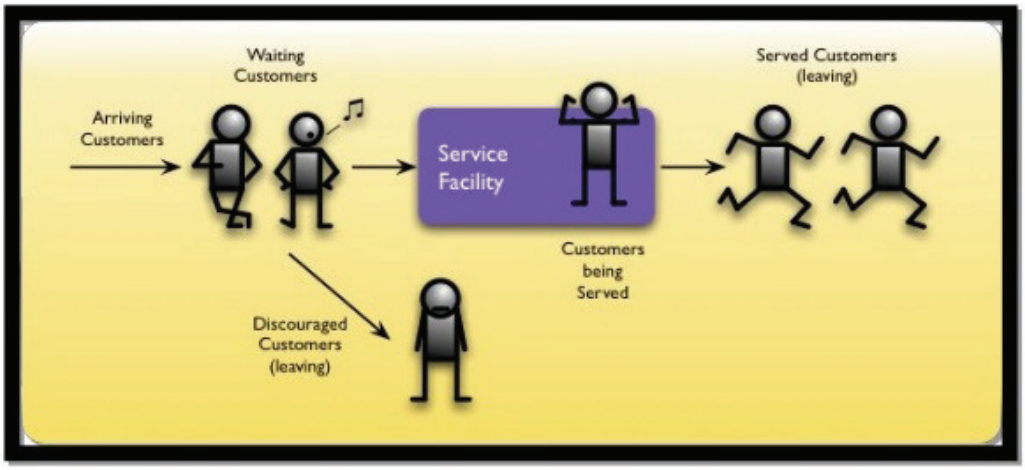

Figure 1.1: Queueing system

1.2 Notations

The queueing process is represented using the standard notation such as $A/B/C/X/Y/Z$, Where 'A' represents the arrival process, and most common arrival process is Poisson process. 'B' denotes the service process in which the random time of processing can be dealt with some probability distribution. The service time distribution can be exponential, Erlang, or general. 'C' denotes the number of servers which could be single or multiple. 'X' represents the waiting capacity, which can be either finite or infinite, whereas 'Y' designates the source capacity, and 'Z' denotes the queue discipline which may be first come first served (FCFS), last come first served (LCFS) or priority basis. For example, the notation $M/M/2/\infty/\infty/FCFS$ denotes a queueing process in which, the arrival process follows Poisson process, the service time distribution is exponential, the number of servers is 2, the waiting capacity is infinite, the source capacity is infinite, the queue

discipline is FCFS. Some other examples are $M^X/M/1$ (customers arrive in group), $M/G^{(a,b)}/1$ (customers are served in batches), $M^X/M^Y/1$ (arrival occurs in groups and service is provided in batches also). The basics of queueing system along with the involved stochastic process is discussed in the books by Medhi [4], Gross et al. [3], Chaudhry and Templeton [2].

1.3 Different type of queues

In numerous real-life circumstances, we find different type of queues some of which are discussed below.

- **Retrial queueing model:** Many queueing systems have the characteristic that arriving customers finding the service area busy must leave it for the time being and move to some virtual place called as 'orbit' and they seek their requests after certain time. This type of queues are known as retrial queues and the most simple textbook example is a person who is making a phone call. If the line is engaged, then his call is not connected but he tries after some random time.

- **Optional service queueing model:** In numerous real-life situations, one may encounter with the queues where all arriving customers first require the essential service and then some of them may require additional service. For instance, all the ships arriving at a port need unloading of items (which is rendered in essential service) but only some of them may require re-loading soon after the unloading which is processed as additional service. Such scenarios can be adequately modeled as queues with optional service.

- **Vacation queueing model:** After finishing off the service of a particular customer, whenever the queue is empty, it leaves for a certain amount of time to perform some other works. This particular time duration is known as vacation period. Such scenarios are framed using vacation queues. After coming back from a vacation, if it finds one or more customers waiting, it initiate the process may be with a set-up time, and then provide service until the system become empty, which is known as single vacation. After coming back from a vacation, if it detect no customer, right away it proceeds for another vacation and continues the same fashion until the arrival of a single customer. This is known as multiple vacation policy. Working vacation is another type of vacation in which the service is rendered at a lower rate during the vacation period rather than absolute termination of the service.

- **Queueing systems with length dependent vacation time:** Queue length dependent vacation time queueing model is a queuing system where the server goes on vacation (or temporarily stops), but its vacation duration depends on how many customers are waiting in the queue. Here the time of vacation is not fixed, rather it depends on the length of the queue. If there are more people in the queue, then the vacation can be shorter (because the demand is more). If there are less people in the queue, then the vacation can be longer (because the workload is less). For example, call centers, manufacturing systems etc.

- **Group arrival queue:** Customers may arrive in groups of arbitrary size in several scenarios, for example family/frends coming to restaurants, bulk arrival of raw material in industries etc. Those can be modeled as group-arrival queue. Here the group-size distribution may follow some particular probability distribution such as geometric, negative binomial etc.

- **Batch service queueing model:** In numerous real-life contexts, it may be observed that the customers are served in batches which have been adequately modeled by batch-service queues. It possess a number of promising applications in diverse areas such as shuttle buses, manufacturing systems, package delivery, chemical processing industries etc. As a classic example of batch-service we can mention also pool-testing policy which is a very handy tool while testing blood samples.

- **Queueing systems with batch-size-dependent service time:** The batch-size-dependent service time queueing model is a queueing system where the service time of a batch of customers depends on the batch size. For example, if a cook prepare food for 10 people, it takes less time as compared to the cooking of food for 100 people. This mechanism is well discussed and employed by Pradhan et al. [5].

While serving the customers in batches one should follow a pre-defined service rule which are discussed below.

- **Fixed batch-size rule:** According to this rule, server serves customers in batches with fixed-batch size say , 'm'.
  Server always serves the batch of size 'm' not more/less than that.

- **General batch-service** (a,b) **rule:** The server initiates the service with a minimum number of customers say, 'a' $(a \geq 1)$. However, for a queue length 'r' where $(a \leq r \leq b)$, the whole queue will be taken into service. For a queue length greater than 'b', the server will serve

the first 'b' customers, and the rest will have to wait for the next round of service.

- **Variable service capacity** $(1,k)$ **rule:** The server initiates the service with a minimum number of customers 1. However, for a queue length 'r' where $(1 \leq r \leq k)$, the whole queue will be taken into service. For a queue length greater than 'k', the server will serve the first 'k' customers, and the rest will have to wait for the next round of service.

1.4 Motivation and objectives

In this project work, a batch-service queue with infinite waiting space is considered mainly due to two reasons: (i) if the waiting space is full, then the arriving customers are not entitled to join the queue which causes losses for the company, and hence the buffer space can be made infinite in many applications by system managers, (ii) batch-service queues have promising applications in group/pool-testing of blood samples to detect viruses, or to find out the defective items in manufacturing companies etc.
Our main objectives are:

- Firstly, to provide a detailed mathematical description of the concerned queue.
- Secondly, the transition probability matrix at service completion epoch has to be constructed.
- To derive the probability generating function of queue length distribution.
- To extract the complete probabilities using roots of the associated denominator.

Chapter 2

Queue-size distribution of a group-arrival queue with set-up time

2.1 Introduction

In the past, the investigation has been taken place for a system where customers arrive individually, and the number of customers the server can handle at once is fixed without considering set-up time. The set-up time is a significate quantity as in many applications we find that some amount of time is always required by the server in configuration before initiation of service. In this chapter, we allow the service time to vary based on the size of the batch being served. This means that larger groups of customers might take more time to be served than smaller ones. We explain the model in detail, construct the transition probability matrix and finally, drive the probability generating function of this model.

2.2 Model description

This group-arrival and batch-size-dependent service queue with set-up time under consideration can be abbreviated as $M^X/G_n^Y/1$.

- **Arrival process:** Customers do not arrive one by one, but instead come in groups or batches. The size of each group (how many customers arrive together) is random. These groups arrive at the system at random times, and the time between group arrivals follows compound Poisson process with rate λ. The probability distribution of arrival batch sizes

is $Pr(X = i) = a_i, \quad i \in \mathbb{N}$, and associated probability generating function $A(z) = \sum_{i=1}^{\infty} a_i z^i$.

- **Service rule:** Customers are served in groups according to 'Y' rule. This rule dictates that customers are served into batches of a randomly determined size Y, which is established by the server at the commencement of service, following a probability mass function $P_r(Y = i)$ for $i = 1, 2, \ldots, B$, where B indicates the server's maximum serving capacity. In instances where the queue is empty, the server enters an idle state, which is terminated upon the arrival of a customer. Whenever, the number is fewer than the chosen capacity 'i', then it does not wait up-to the accumulation of 'i', rather, it processed all the customers awaiting, effectively serving min(i, total queue). The only restriction is that during the ongoing service no one is allowed to join the service, even if free space is there.

- **Service process:** The service time of batches follows the general distribution with density function $S_r(t),\ 1 \leq r \leq B$ and Laplace transform $\bar{S}_r(\theta)$. The service time depends upon the size of the batch being served. This size-dependent service mechanism is used first by Bar-lev et al. [1] for $M/G/1$-type queue.

- **Set-up process:** When there are no customers in the system, the server remains idle and waits for new arrivals. As soon as customer arrives and 'a' have been accumulated, the server begins a set-up phase before it can start serving. This set-up phase takes a random amount of time. Once the setup is completed, the server immediately begins serving a batch of customers.

2.3 Construction of Transition probability matrix and derivation of probability generating function

Let X_{n+1} be the number of customers left behind in the queue after $(n+1)$-th departure and A_{n+1} be the random variable denoting the number of customers that arrive during that service period. A_n^R represent the number of customers arriving during the n^{th} set-up time, and let d be a generic random variable that denotes the number of customers that arriving during an unavailable period.

Therefore we have the following recurrence relation:

$$X_{n+1} = \begin{cases} X_n - Y + A_{n+1}, & \text{if } X_n \geq Y \\ A_{n+1}, & \text{if } 0 < X_n < Y \\ d - Y + A_{n+1}, & \text{if } X_n = 0 \text{ and } d \geq Y \\ A_{n+1}, & \text{if } X_n = 0 \text{ and } d < Y \end{cases} \tag{2.1}$$

The elements of transition probability matrix is denoted by P_{ij}, which are nothing but the one step transition probabilities, and are determined with the help of equation (2.1) and are given by:

$$\begin{aligned} P_{ij} &= \mathbb{P}(X_{n+1} = j \mid X_n = i), \;\; i, j = 0, 1, 2, \ldots \\ &= \begin{cases} \mathbb{P}(X_n - Y + A_{n+1} = j \mid X_n = i), & \text{if } X_n \geq Y \\ \mathbb{P}(A_{n+1} = j \mid X_n = i), & \text{if } 0 < X_n < Y \\ \mathbb{P}(d - Y + A_{n+1} = j \mid X_n = i), & \text{if } X_n = 0 \text{ and } d \geq Y \\ \mathbb{P}(A_{n+1} = j \mid X_n = i), & \text{if } X_n = 0 \text{ and } d < Y \end{cases} \end{aligned}$$

$$= \begin{cases} \mathbb{P}(A_{n+1} = j - i + Y), & \text{if } X_n \geq 0 \\ \mathbb{P}(A_{n+1} = j), & \text{if } 0 < X_n < Y \\ \mathbb{P}(A_{n+1} = j - d + Y), & \text{if } X_n = 0 \text{ and } d \geq Y \\ \mathbb{P}(A_{n+1} = j), & \text{if } X_n = 0 \text{ and } d < Y \end{cases} \tag{2.2}$$

Now the transition probability matrix is constructed and displayed as follows:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ B \\ B+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & & 1 & & 2 & & \cdots \end{array} \\ \begin{bmatrix} \sum_{i=1}^{B} \beta_i \sum_{j=1}^{B} y_j m_0^{(i)} & \sum_{i=1}^{B} \beta_i \sum_{j=1}^{B} y_j m_1^{(i)} + \sum_{i=2}^{B+1} \beta_i y_{i-1} m_0^{(i-1)} & \sum_{i=1}^{B} \beta_i \sum_{j=i}^{B} y_j m_2^{(i)} + \sum_{i=2}^{B+1} \beta_i y_{i-1} m_1^{(i-1)} + \sum_{i=3}^{B+2} \beta_i y_{i-2} m_0^{(i-2)} \sum_{i=2}^{B} y_i m_0^{(2)} & \cdots \\ \sum_{i=1}^{B} y_i m_0^{(1)} & \sum_{i=1}^{B} y_i m_1^{(1)} & \sum_{i=1}^{B} y_i m_2^{(1)} & \cdots \\ \sum_{i=2}^{B} y_i m_0^{(2)} & \sum_{i=2}^{B} y_i m_1^{(2)} + y_1 m_0^{(1)} & \sum_{i=2}^{B} y_i m_2^{(2)} + y_1 m_1^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ y_B m_0^{(B)} & y_B m_1^{(B)} + y_{B-1} m_0^{(B-1)} & y_B m_2^{(B)} + y_{B-1} m_1^{(B-1)} + y_{B-2} m_0^{(B-2)} & \cdots \\ 0 & y_B m_0^{(B)} & y_B m_1^{(B)} + y_{B-1} m_0^{(B-1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

where

$m_i^{(r)}$ = Pr(i arrivals during the service time of a batch of size'r')

α_i = Pr(i arrivals during a setup time) = $Pr(A_n^R = i)$

β_i = Pr(i arrivals during an unavailable period).

Let $\gamma^+ = (\gamma_0^+, \gamma_1^+, \gamma_2^+, \gamma_3^+, \cdots)$ be the stationary probability vector with probability generating function $\Upsilon^+(z) = \sum_{i=0}^{\infty} \gamma_i^+ z^i$, where γ_i^+ represent the probability that n customers are there at departure epoch in a steady-state. Then using ergodicity theorem, we conclude that the probability distribution γ_i^+ does not depend on initial distribution and it is the solution of

$$\gamma^+ = \gamma^+ P, \quad \gamma^+ e = 1 \tag{2.3}$$

where e is the column vector with each quantity as 1.

Now, from equation (2.3), we have the following system of equations:

$$\gamma_0^+ = \gamma_0^+ \sum_{i=1}^{B} \beta_i \sum_{j=i}^{B} y_j m_0^{(i)} + \gamma_1^+ \sum_{i=1}^{B} y_i m_0^{(1)} + \gamma_2^+ \sum_{i=2}^{B} y_i m_0^{(2)} + \cdots + \gamma_B^+ \sum_{i=2}^{B} y_i m_0^{(B)} \tag{2.4}$$

$$\begin{aligned} \gamma_1^+ = {} & \gamma_0^+ \left[\sum_{i=1}^{B} \beta_i \sum_{j=i}^{B} y_j m_1^{(i)} + \sum_{i=2}^{B+1} \beta_i y_{i-1} m_0^{(i-1)} \right] + \gamma_1^+ \sum_{i=1}^{B} y_i m_1^{(1)} + \gamma_2^+ \left[\sum_{i=2}^{B} y_i m_1^{(2)} + y_1 m_0^{(1)} \right] + \cdots \\ & + \gamma_m^+ \left[\sum_{i=m}^{B} y_i m_1^{(m)} + \sum_{i=1}^{m-1} y_i m_{1-i}^{(i)} \right] + \cdots + \gamma_B^+ \left[y_B m_1^{(B)} + y_{B-1} k_0^{(B-1)} \right] + \gamma_{B+1}^+ y_B m_0^{(B)} \end{aligned} \tag{2.5}$$

$$\gamma_2^+ = \gamma_0^+ \left[\sum_{i=1}^{B} \beta_i \sum_{j=i}^{B} y_j m_2^{(i)} + \sum_{i=2}^{B+1} \beta_i y_{i-1} m_1^{(i-1)} + \sum_{i=3}^{B+2} \beta_i y_{i-2} m_0^{(i-2)} \right] + \gamma_1^+ \sum_{i=1}^{B} y_i m_2^{(1)} +$$

$$\gamma_2^+\left[\sum_{i=2}^{B} y_i m_2^{(2)} + y_1 m_1^{(1)}\right] + \cdots + \gamma_m^+\left[\sum_{i=m}^{B} y_i m_2^{(m)} + \sum_{i=1}^{m-1} y_i m_{2-i}^{(i)}\right] + \cdots$$
$$+\gamma_B^+\left[y_B m_2^{(B)} + y_{B-1} m_1^{(B-1)} + y_{B-2} m_0^{(B-2)}\right] + \gamma_{B+1}^+\left[y_B m_1^{(B)} + y_{B-1} m_0^{(B-1)}\right]$$
$$+\gamma_{B+2}^+ y_B m_0^{(B)} \tag{2.6}$$
$$+\ldots$$

In general, we have

$$\begin{aligned}\gamma_n^+ &= \gamma_0^+\left[\sum_{m=1}^{B}\beta_j\sum_{i=m}^{B} y_i m_n^{(m)} + \sum_{m=1}^{B}\beta_m\sum_{i=1}^{m} y_{m-i} m_{n-i}^{(m-i)} + \sum_{m=B+1}^{\infty}\beta_m\sum_{i=1}^{B} y_i m_{n-m+1}^{(i)}\right] \\ &\quad +\gamma_1^+\sum_{i=1}^{B} y_i m_n^{(1)} + \sum_{j=2}^{B}\gamma_j^+\left[\sum_{i=j}^{B} y_i m_n^{(j)} + \sum_{i=\max(1,j-n)}^{j-1} y_i m_{n+i-j}^{(i)}\right] \\ &\quad + \sum_{j=B+1}^{B+n}\gamma_j^+\left[\sum_{i=\max(1,j-n)}^{j-1} y_i m_{n+i-j}^{(i)}\right], \quad n \geq 1 \end{aligned} \tag{2.7}$$

We multiply equations (2.4), (2.5), (2.6), $\cdots$, by z^0, z^1, z^2, $\cdots$, respectively, and adding them, we get the probability generating function as:

$$\Upsilon^+(z) = \frac{\begin{aligned} &z^B\gamma_0\sum_{i=1}^{B}\left[\sum_{j=1}^{i}\beta_j M^{(j)}(z)\right] + \gamma_0\sum_{i=1}^{B} y_i z^{B-i}\left[\beta(z) - \sum_{j=1}^{i}\beta_j z^j - 1\right] M^{(i)}(z) \\ &\qquad + \sum_{j=1}^{B-1}\gamma_j\sum_{i=j}^{B} y_i\left[z^B M^{(j)}(z) - z^{B+j-i} M^{(i)}(z)\right]\end{aligned}}{z^B - \sum_{i=1}^{B} y_i z^{B-i} M^{(i)}(z)} \tag{2.8}$$

where $M^{(r)}(z) = \sum_{j=0}^{\infty} m_j^r z^j = \overline{S}_r(\lambda - \lambda G(z))$

From equation (2.8), our main aim is to acquire the queue-length probabilities. However, it contains some unknown probabilities which to be evaluated first and are discussed below.

2.4 Determination of the unknown quantities in the probability generating function

- There are B unknown quantities $\gamma_0^+, \gamma_1^+, \gamma_2^+, \cdots, \gamma_{B-1}^+$ in the numerator which we need to evaluate.
- By using Rouche's theorem, we conclude that the denominator of the $\Upsilon^+(z)$ say, $F_2(z)$ has

exactly B zeroes in $|z| \leq 1$.

- As we know that the probability generating function in $|z| \leq 1$, the zeroes of $F_2(z)$ must be the zeroes of numerator of $\Upsilon^+(z)$, say $F_1(z)$.
- Let $z_1, z_2, z_3, \cdots z_{B-1}$ be zeroes of $F_2(z)$. Then, $F_1(z_k) = 0$; $k = 1, 2, \cdots, B-1$.
- As we have B unknowns, we need one more equation. This can be obtained using the normalizing condition $\Upsilon^+(1) = 1$.
- By applying L Hospital's rule, we get $\lim_{z\to 1} \frac{F_1'(z)}{F_2'(z)} = 1 \implies \frac{F_1'(1)}{F_2'(1)} = 1$.
- The unknown probabilities $\gamma_0^+, \gamma_1^+, \gamma_2^+, \cdot, \gamma_{B-1}^+$ cab be determined easily by solving the linear simultaneous system of equations.
- Now the function $\Upsilon^+(z)$ is completely known to us which can be inverted either using the Taylor series method or by the method of partial fractions to get all the probability distributions.

2.5 Partial fraction Method

For the known service time distributions, we substitute $M^{(r)}(z) = \overline{S}_r(\lambda - \lambda G(z))$. $1 \leq r \leq B$ in above expression. Let the degrees of $F_1(z)$ and $F_2(z)$ are d_1 and d_2 respectively. Let $\xi_1, \xi_2, \xi_3, \cdots, \xi_{d_2}$ be the total distinct roots of $F_2(z) = 0$. After canceling out B inside roots, we are led to $d_2 - B$ roots in $|z| > 1$. Now we come up with two cases to elaborately discuss the process.

Case 1: $d_1 > d_2$

Applying the partial fraction expansion, the rational function $\Upsilon^+(z)$ can be uniquely written as

$$\Upsilon^+(z) = \sum_{n=0}^{\infty} \gamma_n^+ z^n = \sum_{i=0}^{d_1-d_2} \delta_i z^i + \sum_{k=1}^{d_2-B} \frac{e_k}{\xi_k - z}$$

where δ_i are the resulting coefficient after division of $F_1(z)$ by $F_2(z)$ and e_k are constants.
Using the Residue Theorem, we have

$$e_k = -\frac{F_1(\xi_k)}{F_2'(\xi_k)}, \quad k = 1, 2, 3, \ldots, d_2 - B$$

Now, collecting the coefficient of z^n from both sides of the equation, we have

$$\gamma_n^+ = \delta_n + \sum_{k=1}^{d_2-B} \frac{e_k}{\xi_k^{n+1}}, \quad n \geq 0$$

Case 2: $d_1 < d_2$ Applying the partial fraction technique on $\Upsilon^+(z)$, we can write as

$$\Upsilon^+(z) = \sum_{n=0}^{\infty} \gamma_n^+ z^n = \sum_{k=1}^{d_2-B} \frac{e_k}{\xi_k - z},$$

where

$$e_k = -\frac{F_1(\xi_k)}{F_2'(\xi_k)}, \quad k = 1, 2, 3, \ldots, d_2 - B$$

Now, collecting the coefficient of z^n from both sides of the equation, we get

$$\gamma_n^+ = \sum_{k=1}^{d_2-B} \frac{e_k}{\xi_k^{n+1}}, \quad n \geq 0$$

This completes the extraction process of queue length distribution.

Remark: If the roots of $F_2(z)$ in $|z| > 1$ are multiple, then the partial fraction has to modified accordingly.

2.6 Conclusions and future scopes

In this dissertation, we have studied basics of queueing systems and different type of queues. A group-arrival queue with set-up time, size-dependent service time and random serving capacity rule has been elaborately discussed in which we have constructed transition probability matrix. Then, we have derived the probability generating function of queue-length. Finally, partial fraction has been incorporated to acquire the distribution in terms of roots.

In future, the following can be incorporated in this model:

- Different vacation policies can be included along with queue-length-dependent vacation time.
- Also, server breakdown can also be considered as this happens very often in industries.
- Other batch-service rules may be employed to this model which will lead to different set of new results.

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